Chapter Fifteen

Area Related Theorems and Constructions

We know that bounded plane figures may have different shapes. If the region is bounded by four sides, it is known as quadrilateral. Quadrilaterals have classification and they are also named after their shapes and properties. Apart from these, there are regions bounded by more than four sides. These are polygonal regions or simply polygons. The measurement of a closed region in a plane is known as area of the region. For measurement of areas usually the area of a square with sides of 1 unit of length is used as the unit area and their areas are expressed in square units. For example, the area of Bangladesh is 1.4 lacs square kilometres (approximately). Thus, in our day to day life we need to know and measure areas of polygons for meeting the necessity of life. So, it is important for the learners to have a comprehensive knowledge about areas of polygons. Areas of polygons and related theorems and constructions are presented here.

At the end of the chapter, the students will be able to

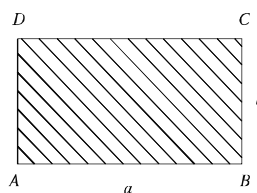
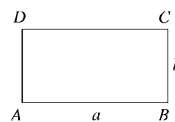
- Explain the area of polygons
- Verify and prove theorems related to areas
- Construct polygons and justify construction by using given data
- Construct a quadrilateral with area equal to the area of a triangle
- Construct a triangle with area equal to the area of a quadrilateral

15·1 Area of a Plane Region

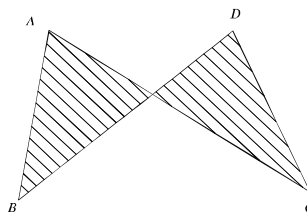
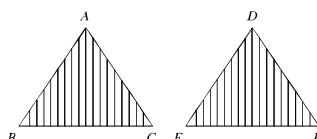
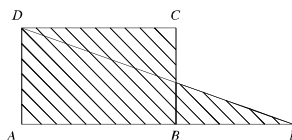
Every closed plane region has definite area. In order to measure such area, usually the area of a square having sides of unit length is taken as the unit. For example, the area of a square with a side of length 1 cm. is 1 square centimetre.

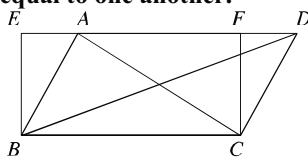
We know that,

(a) In the rectangular region $ABCD$ if the length $AB = a$ units (say, metre), breadth $BC = b$ units (say, metre), the area of the region $ABCD = ab$ square units (say, square metres).

(b) In the square region $ABCD$ if the length of a side $AB = a$ units (say, metre), the area of the region $ABCD = a^2$ square units (say, square metres).

When the area of two regions are equal, the sign '=' is used between them. For example, in the figure the area of the rectangular region *ABCD* = Area of the triangular region *AED*.

It is noted that if ΔABC and ΔDEF are congruent, we write $\Delta ABC \cong \Delta DEF$. In this case, the area of the triangular region *ABC* = area of the triangular region *DEF*.

But, two triangles are not necessarily congruent when they have equal areas. For example, in the figure, area of ΔABC = area of ΔDBC but ΔABC and ΔDBC are not congruent.

Theorem 1

Areas of all the triangular regions having same base and lying between the same pair of parallel lines are equal to one another.

Let the triangular regions *ABC* and *DBC* stand on the same base *BC* and lie between the pair of parallel lines *BC* and *AD*. It is required to prove that, Δ region *ABC* = Δ region *DBC*.

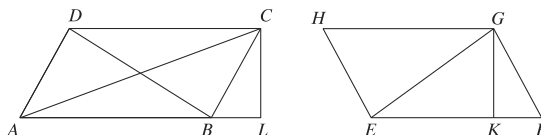
Construction : At the points *B* and *C* of the line segment *BC,* draw perpendiculars *BE* and *CF* respectively. They intersect the line *AD* or *AD* produced at the points *E* and *F* respectively. As a result a rectangular region *EBCF* is formed.

Proof : According to the construction, *EBCF* is a rectangular region. The triangular region *ABC* and rectangular region *EBCF* stand on the same base *BC* and lie between the two parallel line segments *BC* and *ED*. Hence, Δ region *ABC* = $\frac{1}{2}$ (rectangular region *EBCF*)

Similarly, Δ region *DBC* = $\frac{1}{2}$ (rectangular region *EBCF*)

$\therefore$ Δ region *ABC* = Δ region *DBC* (proved).

Theorem 2

Parallelograms lying on the same base and between the same pair of parallel lines are of equal area.

Let the parallelograms regions *ABCD* and *EFGH* stand on the same base and lie between the pair of parallel lines *AF* and *DG* and *AB* = *EF*. It is required to prove that, area of the parallelogram *ABCD* = area of the parallelogram *EFGH.*

Construction :

The base *EF* of *EFGH* is equal. Join *AC* and *EG*. From the points *C* and *G,* draw perpendiculars *CL* and *GK* to the base *AF* respectively.

Proof: The area of $\Delta ABC = \frac{1}{2} AB \times CL$ and the area of ΔEFG is $\frac{1}{2} EF \times GK$.

$\because AB = EF$ and $CL = GK$ (by construction)

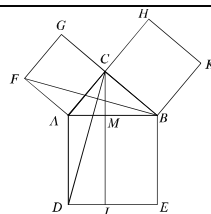
Therefore, area of ΔABC = area of the triangle *EFG*

$\Rightarrow \frac{1}{2}$ area of the parallelogram *ABCD* = $\frac{1}{2}$ area of the parallelogram *EFGH*

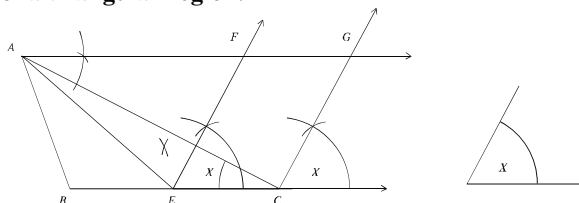
Area of the parallelogram *ABCD* = area of the parallelogram *EFGH*. (Proved)

Theorem 3 (Pythagoras Theorem)

In a right angles triangle, the square of the hypotenuse is equal to the sum of squares of other two sides.

<table>
<tr><td colspan="1">Proposition: Let ABC be a right angled triangle in which $\angle ACB$ is a right angle and hypotenuse is AB. It is to be proved that $AB^2 = BC^2 + AC^2$.
Construction: Draw three squares ABED, ACGF and BCHK on the external sides of AB, AC and BC respectively. Through C, draw the line segment CL parallel to AD which intersects AB and DE at M and L respectively. Join C, D and B, F.
Proof:</td><td>
</td></tr>
<tr><td>Steps</td><td>Justification</td></tr>
<tr><td>(1) In $\Delta\, CAD$ and $\Delta\, FAB$, $CA = AF$, $AD = AB$ and included $\angle CAD = \angle CAB + \angle BAD$
$= \angle CAB + \angle CAF$
= included $\angle BAF$</td><td>[$\angle BAD = \angle CAF$ = 1 right angle]</td></tr>
</table>

Therefore, $\Delta\ CAD \cong \Delta\ FAB$ (2) Triangular region CAD and rectangular region $ADLM$ lie on the same base AD and between the parallel lines AD and CL . Therefore, Rectangular region $ADLM = 2$(triangular region CAD)	[SAS theorem] [Theorem 1]
(3) Triangular region BAF and the square $ACGF$ lie on the same base AF and between the parallel lines AF and BG . Hence Square region $ACGF = 2$(triangular region FAB) = 2(triangular region CAD)	[Theorem 1]
(4) Rectangular region $ADLM$ = square region $ACGF$	[From (2) and (3)]
(5) Similarly joining C, E and A, K , it can be proved that rectangular region $BELM$ = square region $BCHK$	
(6) Rectangular region ($ADLM + BELM$)= square region $ACGF$ + square region $BCHK$ or, square region $ABED$ = square region $ACGF$ + square region $BCHK$ That is, $AB^2 = BC^2 + AC^2$ [Proved]	[From (4) and (5)]

Construction 1

Construct a parallelogram with an angle equal to a definite angle and area equal to that of a triangular region.

Let ABC be a triangular region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle equal to $\angle x$ and area equal to the area of the triangular region ABC.

Construction:

Bisect the line segment BC at E. At the point E of the line segment EC, construct $\angle CEF$ equal to $\angle x$. Through A, construct AG parallel to BC which intersects the ray EF at F. Again, through C, construct the ray CG parallel to EF which intersects the ray AG at G. Hence, $ECGF$ is the required parallelogram.

Proof: Join A, E. Now, area of the triangular region ABE = area of the triangular region AEC [since base BE = base EC and heights of both the triangles are equal]

$\therefore$ area of the triangular region ABC = 2 (area of the triangular region AEC).

Again, area of the parallelogram region *ECGF* is 2 (area of the triangular region *AEC*) [since both lie on the same base and $EC \parallel AG$).

$\therefore$ area of the parallelogram region *ECGF* = area of the triangular region *ABC*.

Again $\angle CEF = \angle x$ [since $EF \parallel CB$ by construction]

So the parallelogram *ECGF* is the required parallelogram.

Construction 2

Construct a triangle with area of the triangular region equal to that of a quadrilateral region.

Let *ABCD* be a quadrilateral region. To construct a triangle such that area of the is triangular region equal to that of a rectangular region *ABCD*.

Construction :

Join *D*, *B*. Through *C,* construct *CE* parallel to *DB* which intersects the side *AB* extended at *E*. Join *D, E*. Then, ΔDAE is the required triangle.

Proof: The triangles *BDC* and *BDE* lie on the same base *BD* and $DB \parallel CE$ (by construction).

$\therefore$ area of the triangular region *BDC* = area of the triangular region *BDE*.

$\therefore$ area of the triangular region *BDC* + area of the triangular region *ABD* = area of the triangular region *BDE* + area of the triangular region *ABD*

$\therefore$ area of the quadrilateral region *ABCD* = area of the triangular region *ADE* .

Therefore, ΔADE is the required triangle.

N.B. Applying the above mentioned method innumerable numbers of triangles can be drawn whose area is equal to the area of a given quadrilateral region.

Construction 3

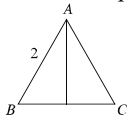
Construct a parallelogram, with a given an angle and the area of the bounded region equal to that of a quadrilateral region.

Let *ABCD* be a quadrilateral region and $\angle x$ be a definite angle. It is required to construct a parallelogram with angle $\angle x$ and the area equal to area of the quadrilateral region *ABCD*.

Construction:
Join B, D. Through C construct CF parallel to DB which intersects the side AB extended at F. Find the midpoint G of the line segment AF. At A of the line segment AG, construct $\angle GAK$ equal to $\angle x$ and draw $GH \parallel AK$ through G. Again draw $KDH \parallel AG$ through D which intersects AK and GH at K and H respectively.
Hence $AGHK$ is the required parallelogram.
Proof: Join D, E. By construction $AGHK$ is a parallelogram.
where $\angle GAK = \angle x$. Again, area of the triangular region DAF = area of the rectangular region $ABCD$ and area of the parallelogram $AGHK$ = area of the triangular region DAF. Therefore, $AGHK$ is the required parallelogram.

Exercise 15

1. The lengths of three sides of a triangle are given, in which case below the construction of the right angled triangle is not possible ?
   (a) 3 cm, 4 cm, 5 cm (b) 6 cm, 8 cm, 10 cm
   (c) 5 cm, 7 cm, 9 cm (d) 5 cm, 12 cm, 13 cm
2. Observe the following information :
   i. Each of the bounded plane has definite area.
   ii. If the area of two triangles is equal, the two angles are congruent.
   iii. If the two angles are congruent, their area is equal.

Which one of the following is correct ?
   (a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii
   Answer question no. 3 and 4 on the basis of the information in the figure below, ΔABC is equilateral, $AD \perp BC$ and $AB = 2$:

3. BD = What ?
   (a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4
4. What is the height of the triangle ?
   (a) $\frac{4}{\sqrt{3}}$ sq. unit (b) $\sqrt{3}$ sq. unit (c) $\frac{2}{\sqrt{3}}$ sq. unit (d) $2\sqrt{3}$ sq. unit.
5. Prove that the diagonals of a parallelo gram divide the parallelogram into four equal triangular regions.
6. Prove that the area of a square is half the area of the square drawn on its diagonal.

7. Prove that any median of a triangle divides the triangular region into two regions of equal area.

8. A parallelogram and a rectangular region of equal area lie on the same side of the bases. Show that the perimeter of the parallelogram is greater than that of the rectangle.

9. *X* and *Y* are the mid points of the sides *AB* and *AC* of the triangle *ABC*. Prove that the area of the triangular region $AXY = \frac{1}{4}$ (area of the triangular region *ABC*)

10. In the figure, *ABCD* is a trapezium with sides *AB* and *CD* parallel. Find the area of the region bounded by the trapezium *ABCD*.

11. *P* is any point interior to the parallelogram *ABCD*. Prove that the area of the triangular region *PAB* + the area of the triangular region $PCD = \frac{1}{2}$ (area of the parallelogram *ABCD*).

12. A line parallel to *BC* of the triangle *ABC* intersects *AB* and *AC* at *D* and *E* respectively. Prove that the area of the triangular region *DBC* = area of the triangular region *EBC* and area of the triangular region *DBF* = area of the triangular region *CDE*.

13. $\angle A = 1$ right angle of the triangle *ABC*. *D* is a point on *AC*. Prove that $BC^2 + AD^2 = BD^2 + AC^2$.

14. *ABC* is an equilateral triangle and *AD* is perpendicular to *BC*. Prove that $4AD^2 + 3AB^2$.

15. *ABC* is an isosceles triangle. *BC* is its hypotenuse and *P* is any point on *BC*. Prove that $PB^2 + PC^2 = 2PA^2$.

16. *C* is an obtuse angle of ΔABC; *AD* is a perpendicular to *BC*. Show that.
$AB^2 = AC^2 + BC^2 + 2BC.CD$

17. *C* is an acute angle of ΔABC ; *AD* is a perpendicular to *BC*. Show that $AB^2 = AC^2 + BC^2 - 2BC.CD$.

18. *AD* is a median of ΔABC. Show that, $AB^2 + AC^2 = 2(BD^2 + AD^2)$.